2.3 Application to Spanning and Linear Independence

Quote. "God exists since mathematics is consistent, and the devil exists since we cannot prove the consistency" Morris Kline (1908-1992)

Vocabulary.

- Dimension: the number of linearly independent vectors required to ~~space~~ span the space.

1. **Purpose of this section**

   There will not be anything new in this section. By carefully going through some selected examples, we will see how to use Gaussian elimination and the rank calculation to answer some of the questions we earlier saw about spanning sets, linear independence, and bases.

2. **Determining if a given vector is in the span of a set of other vectors.**

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\} \qquad \vec{v}_1 = \begin{pmatrix} 5 \\ 4 \\ 6 \\ 7 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

Is $\boxed{\vec{v}_1}$ or $\vec{v}_2$ in span(B)?

$$\begin{pmatrix} 5 \\ 4 \\ 6 \\ 7 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

inconsistent

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 2 & 6 \\ 1 & 1 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow -x_3 = 3 \\ \leftarrow 3x_3 = 6 \end{array}$$

$\vec{v}_1$ is NOT in the span of B

$$\vec{v}_2 = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \boxed{0} & \boxed{0} \end{array}\right] \end{array} \begin{array}{l} \\ \leftarrow c_1 = 3 \\ \leftarrow c_2 = -1 \\ \leftarrow c_3 = -1 \\ \\ \end{array}$$

$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

consistent, 0 free variables rank=3, n=3

$\vec{v}_2 \in \text{span}(\beta)$ $\vec{v}_2 = 3\vec{b}_1 - \vec{b}_2 - \vec{b}_3$

3. **Taking a space or subspace defined as a span of some vectors and identifying the space and the system of linear equations that defines the space.**

Find a system for which

$$\underbrace{\text{span}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}}$$ in the solution set.

Yes, spans are always subspace

- always contains $\vec{0}$

$$A\vec{x} = 0 \qquad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = t\begin{pmatrix}2\\1\\-1\end{pmatrix} + s\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

unknowns

$$\left[\begin{array}{cc|c} 2 & 1 & x_1 \\ 1 & -2 & x_2 \\ -1 & 1 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & x_2 \\ 2 & 1 & x_1 \\ -1 & 1 & x_3 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & -2 & x_2 \\ 0 & 1 & 1/5(x_1 - 2x_2) \\ 0 & 0 & 1/5x_1 + 3/5x_2 + x_3 \end{array}\right]$$

$$1/5x_1 + 3/5x_2 + x_3 = 0$$

$$\boxed{x_1 + 3x_2 + 5x_3 = 0}$$

(Plane)

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \end{array}\right]$$

$$y_1 = m_1x_1 + b_1$$

(don't know why it's b, always b)

$$y_1 = m_2x_1 + b_2$$

√ $x_1 + 2x_2 = 3$

√ $x_1 + x_2 = 5$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 1 & 5 \end{array}\right]$$

$A \quad \vec{b}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

$1x_1 + 2x_2 = 3$

$-x_1 + x_2 = 5$

4. **Using Gaussian Elimination to determine linear independence.**

Are $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Are there non-zero c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The set in linearly DEPENDENT

$$\boxed{\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

rank=2
n=3   1 free variable

consistent;

5. **Using rank to determine linear independence and whether or not a set of vectors forms a basis.**

Ex. Is $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$

- No, we need at least 3
-

- Flatland
- Edward Abbey

$$\begin{bmatrix} -2 & 3 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ vectors are linearly independent

$A_{3\times 2}$ rank=2

Ex. Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ has a basis?

- Is there 3 vectors?

(This might be a basis)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$A_{3\times\boxed{3}}$ rank=3

$\therefore$ Linear Independence, is a basis

(We have to check rank, not necessarily RREF)

6. **Dimension**

The **dimension** of a space is the number of vectors in the basis for that space. The dimension of the trivial subspace (the zero vector) is zero.

If $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis

then dimension of the space spanned $=2$

and it is in $\mathbb{R}^2$

span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$
- dimension is 2
- are in $\mathbb{R}^3$

Example.

Find the basis and dimension of the solution space of the following system of homogeneous linear equations:

$$\begin{aligned} x_1+3x_2+2x_3&=0\\ 2x_1+8x_2+6x_3&=0\\ -4x_1+\ x_2+3x_3&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0\\ 2 & 8 & 6 & 0\\ -4 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & -4 & 0 \end{array}\right]$$

$x_3=0$

$x_2=0$

$x_1=0$

Solution space $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

dimension is 0

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

dimension $=1$

$\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix}$ $\quad$ $\vec{x} = t\begin{bmatrix}0\\0\end{bmatrix}$