2.2 Reduced Row Echelon Form, Rank and Homogenous Systems

Quote. "Our mathematical models of physical reality are far from complete, but they provide us with schemes that model reality with great precision - a precision enormously exceeding that of any description free of mathematics" Roger Penrose (1931-)

Vocabulary.

- leading entry: the first non-zero entry in a row
- Gaussian elimination: used to put a matrix into row echelon form.
- Gauss-Jordon elimination: used to put a matrix into reduced row echelon form
- REF: row echelon form.
- RREF: reduced row echelon form.
- row equivalent: two matrices which have the same solution set.
- rank (of a matrix): the number leading 1's in the RREF form of the matrix.

1. **Gauss-Jordan elimination**

   Purpose: Converts an augmented matrix to ***reduced*** *row echelon form.*

   - **Forward Phase.** *Gauss elimination*
   - **Backward Phase.** Beginning with the last nonzero row and working *upward*, add suitable multiples of each row to the rows above to introduce zeros above the *leading 1*'s.

   Example. Perform the backward phase on the matrix obtained from the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. **Some facts about row echelon forms**

   First a theorem:

   - Every matrix has a **unique** *reduced row echelon form.*
   - *Row echelon forms* are not unique. But all *row echelon forms* have **leading 1**'s in the same positions of the matrix.

   And then two definitions:

   - **Pivot positions/columns** The positions in a row echelon form that have the *leading 1*'s are called **pivot positions**.
   - The columns that contain the *leading 1*'s are called **pivot columns**.

   And a question:

   What's the correspondence between pivot columns and the leading and free variables?

3. **Solving linear systems - summary**

4. **Rank**

Definition. The rank of a matrix A is the number of leading ones in its reduced row echelon form and is denoted by rank(A).

Example. What is the rank of

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) = 3

iff

5. **Using Rank to determine consistency**

A linear system is consistent if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix.

Example.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$x_1 + 2x_2 = 1$

$0x_1 + x_2 = 3$

consistent, unique

Both rank 2 so ⇒ unique soln

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Both rank 1 so ⇒ consistent, infinite, x_2 = free

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank = 1 rank = 2

6. **Using Rank to determine the number of free variables**

Given a consistent system then the number of free variables (or parameters) in the solutions is the number of variables minus the rank of the matrix A.

Example.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

rank = 2

n = 2

free variables = 0

n - rank = 0

A_{2x2} A_{mxn} ← columns, ↑ rows

b

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

rank = 1

n = 2

# of free variables

2 - 1 = 1

7. **Homogeneous linear systems**

- A linear system is called **homogeneous** if each of its equations is homogeneous. (This means that the last column of the augmented matrix consists only of zeros.)

$x+2y=0$
$2x-y=0$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right]$

- Every homogeneous linear system has at least one solution, called the **trivial** solution:

$\vec{x}=\vec{0}=\begin{bmatrix}0\\0\end{bmatrix}$

- Any other solutions, if they exist, are called **nontrivial** solutions.

0
1
2 → ∞

- Note that if a homogeneous system has some nontrivial solution

$$x_1 = s_1, \quad x_2 = s_2, \quad \ldots, x_n = s_n,$$

where $s_1, s_2, \ldots, s_n$ are some numbers, then it must have infinitely many solutions since

$$x_1 = ks_1, \quad x_2 = ks_2, \quad \ldots, x_n = ks_n,$$

is also a solution for any scalar k.

- **Theorem** A *homogeneous* linear system has only the *trivial* solution or it has *infinitely many* solutions.

Since a homogeneous linear system always has a solution, we cannot have a row with zeros everywhere except for the last column in its RREF, i.e., rows of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

Equivalently, each of the nonzero rows in its RREF contains a *leading* variable.

8. **Solution space**

The solution set of a **homogeneous** system is a **subspace.** This subspace is called the **solution space**.

$A\vec{x}=0$ ← solution in subspace
← called situation space

$A\vec{x}=\vec{b}$, $\vec{b}\neq 0$ ← solution does NOT contain origin...