# 2.2 Reduced Row Echelon Form, Rank and Homogenous Systems

**Quote.** "Our mathematical models of physical reality are far from complete, but they provide us with schemes that model reality with great precision - a precision enormously exceeding that of any description free of mathematics" Roger Penrose (1931-)

## Vocabulary.

- leading entry: the first non-zero entry in a row
- Gaussian elimination: used to put a matrix into row echelon form.
- Gauss-Jordon elimination: used to put a matrix into reduced row echelon form
- REF: row echelon form.
- RREF: reduced row echelon form.
- row equivalent: two matrices which have the same solution set.
- rank (of a matrix): the number leading 1's in the RREF form of the matrix.

#### 1. Gauss-Jordan elimination

**Purpose:** Converts an augmented matrix to *reduced row echelon form*.

- Forward Phase. Gauss elimination
- Backward Phase. Beginning with the last nonzero row and working *upward*, add suitable multiples of each row to the rows above to introduce zeros above the *leading* 1's.

**Example.** Perform the backward phase on the matrix obtained from the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### 2. Some facts about row echelon forms

First a theorem:

- Every matrix has a **unique** reduced row echelon form.
- Row echelon forms are not unique. But all row echelon forms have **leading 1**'s in the same positions of the matrix.

And then two definitions:

- **Pivot positions/columns** The positions in a row echelon form that have the *leading* 1's are called **pivot positions**.
- The columns that contain the *leading 1*'s are called **pivot columns**.

And a question:

What's the correspondence between pivot columns and the leading and free variables?

3. Solving linear systems - summary

### 4. Rank

**Definition.** The rank of a matrix A is the number of leading ones in its reduced row echelon form and is denoted by rank(A).

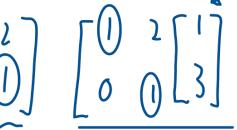
Example. What is the rank of

# iff

# 5. Using Rank to determine consistency

A linear system is consistent if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix.

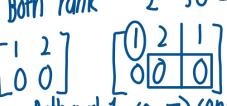
Example.





consistent,

Both rank  $\frac{1}{2}$  so => unique sol n



2 | O]] => consistent, infinite, 1,2= free

# 6. Using Rank to determine the number of free variables

Given a consistent system then the number of free variables (or parameters) in the solutions is the number of variables minus the rank of the matrix A.

Example.



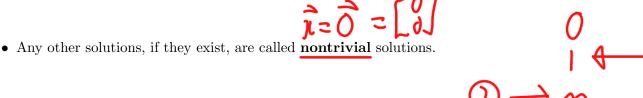
free variables =0

# 7. Homogeneous linear systems

• A linear system is called **homogeneous** if each of its equations is homogeneous.

(This means that the last column of the augmented matrix consists only of zeros.)

• Every homogeneous linear system has at least one solution, called the **trivial** solution:



• Note that if a homogeneous system has some nontrivial solution

$$x_1 = s_1, \quad x_2 = s_2, \quad \dots, x_n = s_n,$$

where  $s_1, s_2, \ldots, s_n$  are some numbers, then it must have infinitely many solutions since

$$x_1 = ks_1, \quad x_2 = ks_2, \quad \dots, x_n = ks_n,$$

is also a solution for any scalar k.

• Theorem A homogeneous linear system has only the trivial solution or it has infinitely many solutions.

Since a homogeneous linear system always has a solution, we cannot have a row with zeros everywhere except for the last column in its RREF, i.e., rows of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

Equivalently, each of the nonzero rows in its RREF contains a leading variable.

## 8. Solution space

The solution set of a **homogeneous** system is a **subspace**. This subspace is called the **solution space**.

$$Ax = 0 \leftarrow$$
 solution in subspace called situation space  $Ax = b$ ,  $b \neq 0 \leftarrow$  solution does NOT contain origin...