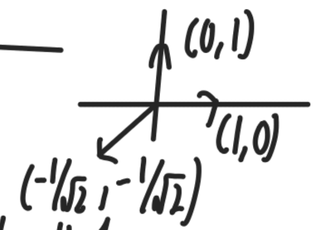
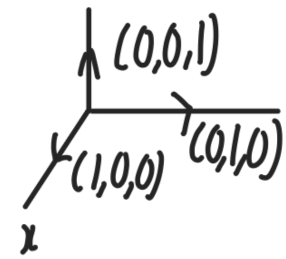
1.2 Span and Linear Independence

Quote. "Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost." Alexander Grothendieck (1928-2014)

Vocabulary.

- trivial solution: the solution that consists of the zero vector.
- unit vector: a vector of length 1.
- $\boxed{\text{span:}}$ this is ____________ and is defined in detail below.
- standard ______ vectors in $\mathbb{R}^2$: $\{\mathbf{e_1} = (1,0), \mathbf{e_2} = (0,1)\}$ always length 1
- standard ______ vectors in $\mathbb{R}^3$: $\{\mathbf{e_1} = (1,0,0), \mathbf{e_2} = (0,1,0), \mathbf{e_3} = (0,0,1)\}$
- $\boxed{\text{linear independence:}}$ this is VERY important and is defined in detail below.
- basis: a minimum spanning set.



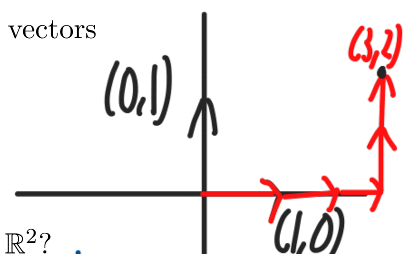


1. **Definition:** Span:

If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the set of all linear combinations of these vectors

$$\{\mathbf{x};\ \mathbf{x} = t_1\mathbf{v_1} + t_2\mathbf{v_2} + \cdots + t_s\mathbf{v_s},\ t_1, t_2, \ldots, t_s \in \mathbb{R}\}$$

is called the **span** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}$ and denoted by span$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}\}$.



What is the set $W_1 = \text{span}\{\mathbf{e_1}, \mathbf{e_2}\}$, where $\mathbf{e_1}, \mathbf{e_2}$ are the standard basis vectors in $\mathbb{R}^2$?

$\vec{e}_1 = (1,0)$ $\vec{e}_2 = (0,1)$

in $\mathbb{R}^2$ x-axis $= \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$

What is the set $W_2 = \text{span}\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are the standard basis vectors in $\mathbb{R}^3$?

$\vec{e}_1 = (1,0,0)$ $\vec{e}_3 = (0,0,1)$
$\vec{e}_2 = (0,1,0)$

$\begin{bmatrix}3\\2\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$

What is the set $W_3 = \text{span}\{(-1,4)\}$?

$W_3 = \left\{ t\begin{bmatrix}-1\\4\end{bmatrix} \mid t \in \mathbb{R} \right\}$

What is the set $W_4 = \text{span}\{(1,0,5),(0,1,5)\}$?

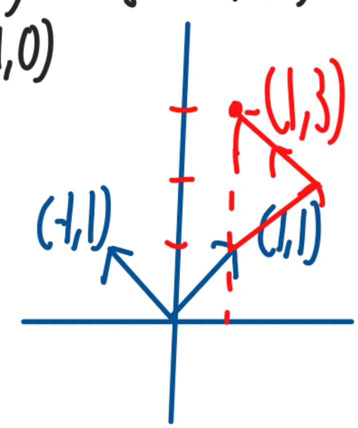
z

$\boxed{t}\begin{bmatrix}1\\0\\5\end{bmatrix} / + \boxed{s}\begin{bmatrix}0\\1\\5\end{bmatrix}$

•plane

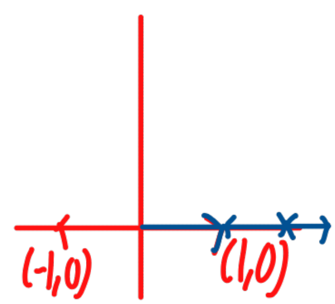
$\begin{bmatrix}1\\0\\5\end{bmatrix} \neq m\begin{bmatrix}0\\1\\5\end{bmatrix}$

$W_5 = \text{span}\{(1,0,5),(10,0,50)\}$. line



$\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

$\begin{bmatrix}1\\3\end{bmatrix} = t\begin{bmatrix}1\\1\end{bmatrix} + s\begin{bmatrix}-1\\1\end{bmatrix}$

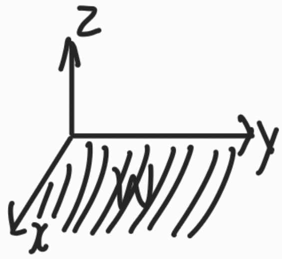


$\mathbb{R}^1 = \{[0], [1], [\pi]\}$

- understand plane and parametric curves
- understand def'ns of words!

Investigate set W for diff vectors v_1, v_2

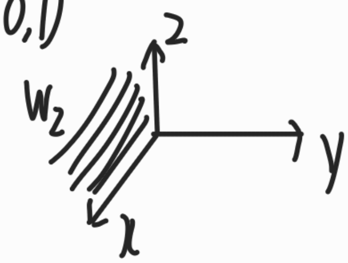
$\vec{v_1} = (1,0,0)$ $\vec{v_2} = \underline{(0,1,0)}$



$W = span(\vec{v_1}, \vec{v_2})$

$\vec{v_1} = (1,0,0)$ $\vec{v_2} = (0,0,1)$

$W_2 = span(\vec{v_1}, \vec{v_2})$



2. **Linear independence**

Suppose $\mathbf{v_1}$ and $\mathbf{v_2}$ are vectors in $\mathbb{R}^3$. Consider the set

$$W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\} = \{s\mathbf{v_1} + t\mathbf{v_2};\ s, t \in \mathbb{R}\}$$

Investigate the set W for different vectors $\mathbf{v_1}$ and $\mathbf{v_2}$.

Definition: Linear independence.

A non-empty set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ (or as we will see in $\mathbb{R}^n$) is **linearly independent** if the only scalars $c_1, c_2, \dots, c_k$ that satisfy

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

$$-c_k\vec{v}_k = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{k-1}\vec{v}_{k-1}$$

are $c_1 = 0,\ c_2 = 0,\ \dots,\ c_k = 0.$ (multiply everything by 0)

If there are constants $c_1, c_2, \dots, c_k$, *not all zero*, that satisfy this equation, the set of vectors S is **linearly dependent**.

Are the following sets linearly independent?

(a) $S = \{(1,0,0), (0,0,1)\}$

Yes

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \underbrace{c_1}_{=0}\begin{bmatrix}1\\0\\0\end{bmatrix} + \underbrace{c_2}_{=0}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

(b) $S = \{(0,1,0), (0,3,0)\}$

No $\quad \begin{bmatrix}0\\3\\0\end{bmatrix} = 3\begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\3\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Suppose $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors. Are the following sets linearly independent?

(a) $\{\mathbf{u}\}$ Yes

(b) $\{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$ No, contains a 0 vector

$$c_1\vec{w} + c_2\vec{v} + c_3\vec{0} = \vec{0}$$

$$c_1 = c_2 = 0$$

$$c_3 = 10$$

Theorem 0.1 *A set S with two or more vectors in $\mathbb{R}^n$ is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

Proof:

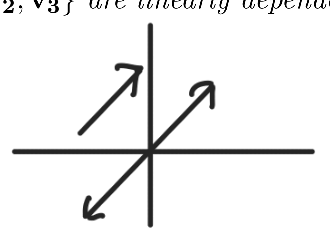
Video

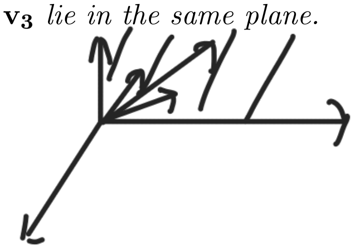
Corollary 0.2

(follow directly)

(a) $\{\mathbf{v_1}, \mathbf{v_2}\}$ *are linearly dependent if and only if* $\mathbf{v_1}$ *and* $\mathbf{v_2}$ *are parallel;*

(b) $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *are linearly dependent if and only if* $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ *lie in the same plane.*





3. **Checking linear independence**

Are the vectors $(5, 4), (-1, 6)$ linearly independent?

$$\boxed{c_1\begin{bmatrix}5\\4\end{bmatrix}} + c_2\begin{bmatrix}-1\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \begin{bmatrix}5c_1\\4c_1\end{bmatrix} + \begin{bmatrix}-c_2\\6c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

① $5c_1 - c_2 = 0$

② $4c_1 + 6c_2 = 0$

$c_2 = 5c_1$

$4c_1 + 6(5c_1) = 0 \Rightarrow c_1 = 0$, $c_2 = 0$

Linearly Independent

Are the vectors $(1, 0, 1), (-1, 0, 1), (2, 0, 2)$ linearly independent? No

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_3\begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$c_1 - c_2 + 2c_3 = 0$

$0c_1 + 0c_2 + 0c_3 = 0$

② $c_1 + c_2 + 2c_3 = 0$

$2c_1 + 4c_3 = 0$

$c_1 = -2c_3$

$c_3 = 1$

$\boxed{c_2 = 0}$

$c_1 = -2c_3$

$c_3 = -5$

$c_1 = 10$

What do the above vectors span?

$$\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \cancel{\begin{pmatrix}2\\0\\2\end{pmatrix}}\right\}$$

$$= \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

plane. through the origin

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Can we express one of the vectors as a linear combination of the others?

$$\begin{bmatrix}2\\0\\2\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

And we imagine that we may need a more efficient way to do this! (coming in section 2.1)

4. Definition: **basis**

A set of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_s}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called a **basis** for $\mathbb{R}^2$ or $\mathbb{R}^3$ if

(a) $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_s}\} = \mathbb{R}^2$ or $\mathbb{R}^3$; and

(b) The set $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_s}\}$ is *linearly independent.*

BOTH

$\vec{e}_1 = (1,0)$

$\vec{e}_2 = (0,1)$

Prove that the set $\{(-1,2),(1,5)\}$ is a basis for $\mathbb{R}^2$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$x = -c_1 + c_2$

$y = 2c_1 + 5c_2$

span

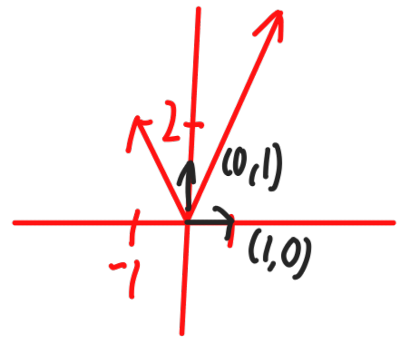
$c_1 = 1/7[-5x + y]$

$c_2 = 1/7(2x + y)$

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear Independence

Yes



Consider the vector $(3,4)$ in relation to the standard basis and in relation to the basis above.

standard basis

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = -11/7 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 10/7 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$3 = -c_1 + c_2 \qquad c_1 = -11/7$

$4 = 2c_1 + 5c_2 \qquad c_2 = 10/7$

Q: # of vectors, L.I., can they NOT space $\mathbb{R}^2$ or $\mathbb{R}^3$)

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

•always show up on problems