$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$ • not basis of $\mathbb{R}_3$

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\mathbb{R}^2$ vectors have 2 components

$\mathbb{R}^3$ vectors have 3

$\mathbb{R}^2$ DNE $\mathbb{R}^3$

$\mathbb{R}^3$ DNE $\mathbb{R}^2$

• Quiz on 1.1, 1.2

# 1.3 Length and Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$

**Quote.** "Mathematics as a science commenced when first someone, probably a Greek, proved propositions about 'any' things or 'some' things, without specifications of definite particular things"

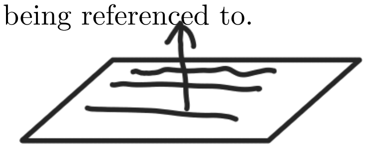
**Vocabulary.**

- length (or magnitude or norm) of a vector: how long the vector is.
- angle of a vector: the angle between the $x$-axis and the vector, measured counterclockwise.


- product: the product of two things is the result of multiplying them.
- dot product: a type of vector multiplication (more below).
- cross product: a type of vector multiplication used only in $\mathbb{R}^3$ (we will not cover this). – be careful when using it
- orthogonal: at right angles, perpendicular. $\perp$
- normal vector: a vector that is orthogonal to what it is being referenced to.

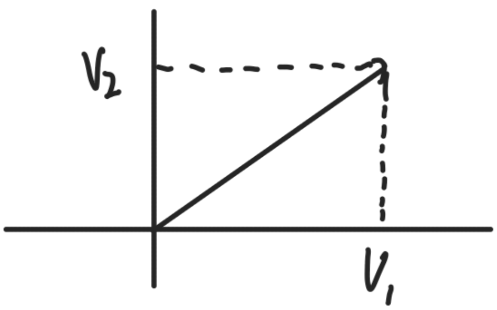
won't use it, but in text useful in $\mathbb{R}^3$

quaternions
↳ will not use



1. **Length of a vector in $\mathbb{R}^2$**

What is the length of a vector $\mathbf{v} = (v_1, v_2)$?




$\sqrt{v_1^2 + v_2^2}$

Definition: The **length** or **magnitude** or **norm** of a vector. The length of a vector $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$ is

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$

norm

$\wedge\, \bar{v}$

**Properties:**

Let $\mathbf{v}$ be a vector and $k$ a scalar. Then

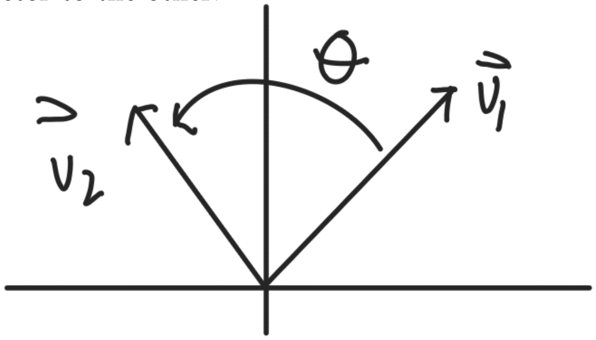
(a) $||\mathbf{v}|| \geq 0$

(b) $||k\mathbf{v}|| = |k|\,||\mathbf{v}||$ mult vector =2x long

(c) $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

2. **Angle between vectors in $\mathbb{R}^2$**

The **angle** between two nonzero vectors is the smallest non-negative angle needed to rotate one vector to the other.




**Theorem 0.1** *If $\alpha$ is the angle between nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in $\mathbb{R}^2$, then*

$$\cos\alpha = \frac{u_1v_1 + u_2v_2}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$$

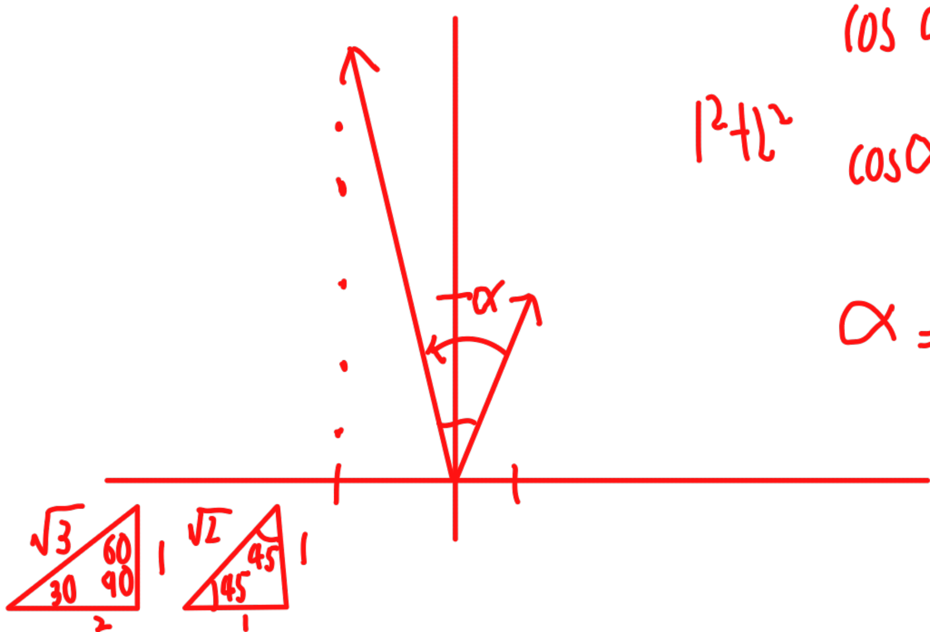
product of the norms

Proof: (Recall the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\alpha$)

Video

Scalar multiplication

**Example.** Find the angle between (1,2) and (-1,5).




$1^2+2^2$

$$\cos\alpha = \frac{u_1v_1 + u_2v_2}{||\vec{u}|| \cdot ||\vec{v}||}$$

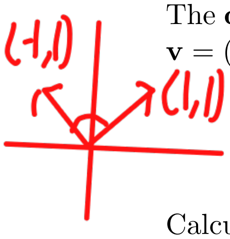
$$\cos\alpha = \frac{(1)(-1)+(2)(5)}{(\sqrt{5})(\sqrt{26})}$$

$$\alpha = \arccos\left(\frac{9}{\sqrt{5}\sqrt{26}}\right)$$

3. **Dot product in $\mathbb{R}^2$**

$\vec{u} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

The **dot product** (also called the **(Euclidean) inner product**) of vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is the *scalar*




$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

"u dot v"

$(1,1)\cdot(-1,1) = 0$ $\quad \cos\alpha = 0$ (orthogonal)

$\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\vec{u}\cdot\vec{v} = (1)(4) + (-1)(5)$

$= -1$

Calculate $\mathbf{u} \cdot \mathbf{u} =$

$u_1 u_1 + u_2 u_2 = u_1^2 + u_2^2$

$\vec{u}\cdot\vec{u} = \|\vec{v}\|^2$

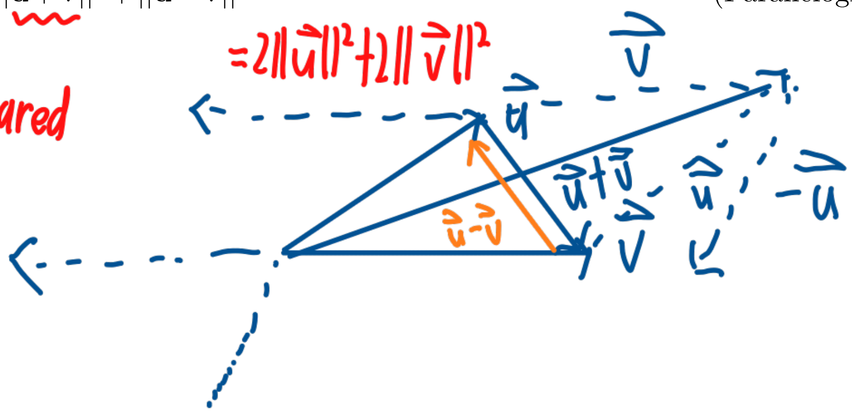
$u = (u_1, u_2)$

$\vec{v} = (v_1, v_2)$

Calculate $||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 =$ (Parallelogram Equality).

$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

Take norm, squared




Algebraic properties

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative

(b) (a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ and (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ $\leftarrow$ distribution

(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(d) $\mathbf{0} \cdot \mathbf{v} = 0$

(e) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ $\leftarrow$

Prove 2(a) using the definition of dot product

$\vec{u} = (1,2,3)$

$\vec{v} = (4,5,6)$

$\vec{u}\cdot\vec{v} = (1)(4) + (2)(5) + (3)(6)$

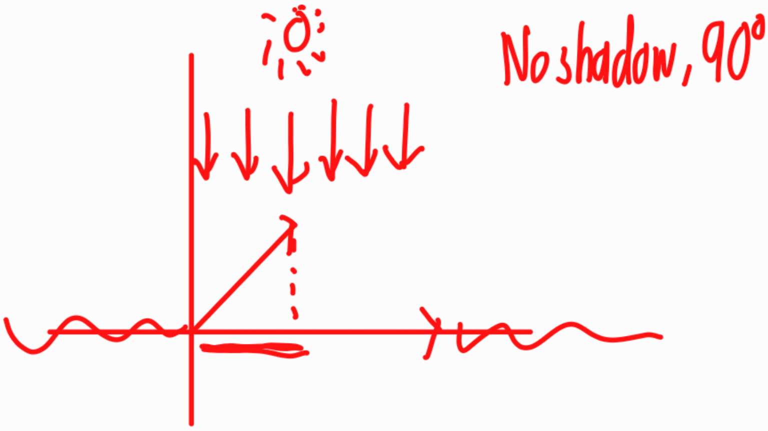
$= 32$

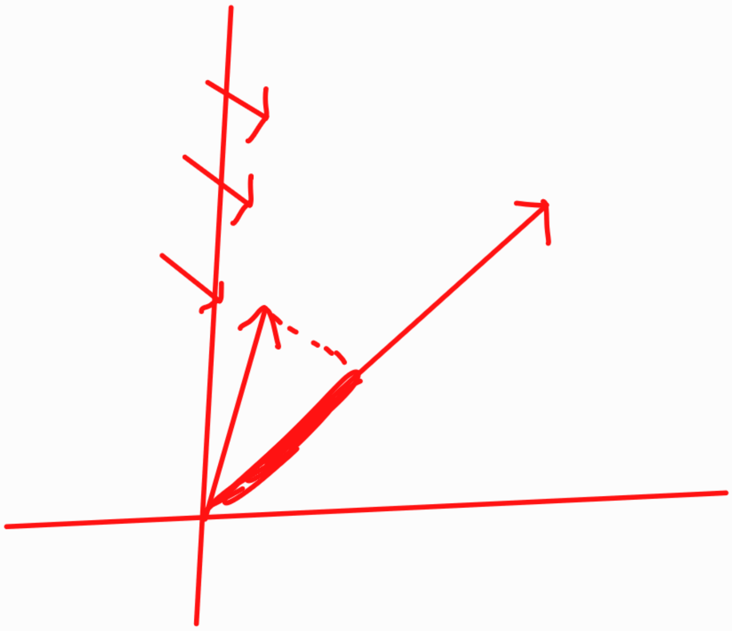
4. **Orthogonality**

Two vectors are **orthogonal** to one another if their dot product is zero.

**Example.**

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$


No shadow, 90°



5. **Length, Angles and Dot products in $\mathbb{R}^3$**

The **dot product** of vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the *scalar*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

**Example.**

Length can be computed using the dot product.

The **length** of the vector $\mathbf{u} = (u_1, u_2, u_3)$ is denoted $||\mathbf{u}||$ and can be computed via $\sqrt{u_1^2 + u_2^2 + u_3^2}$ which is $\sqrt{\mathbf{u} \cdot \mathbf{u}}$.

**Example.**

$\vec{u} = (1, 4, -2)$ $\quad \|\vec{u}\| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{21}$

(not a proof) $\quad \vec{u} \cdot \vec{u} = (1)(1) + (4)(4) + (-2)(-2) = 21 \qquad \|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

6. **Distance**

**Distance between points** $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ (or $\mathbb{R}^2$, with two component vectors), denoted by $d(\mathbf{u}, \mathbf{v})$ is the norm of the vector $\mathbf{v} - \mathbf{u}$, i.e.,

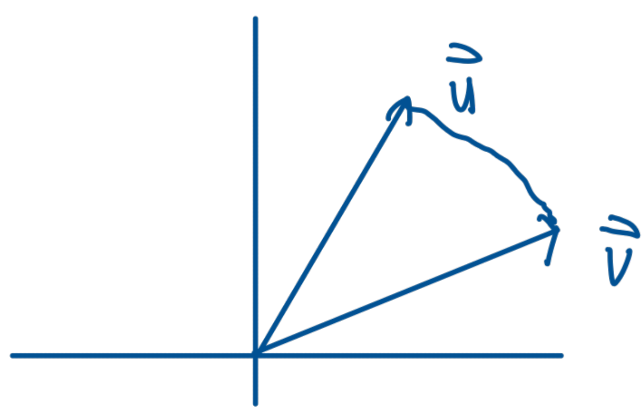
$d(\mathbf{u}, \mathbf{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}$

**Properties:**

(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$

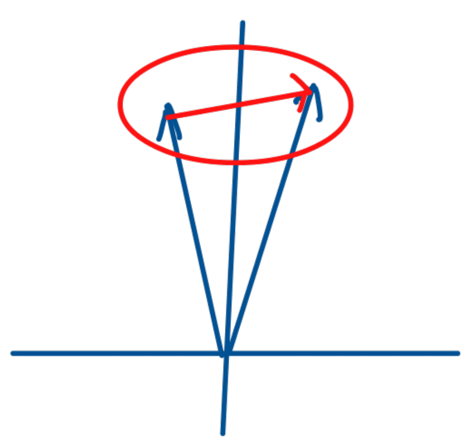
(b) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$



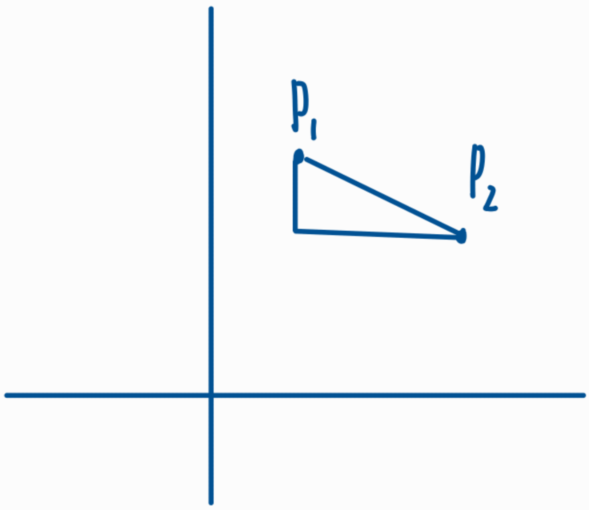

**Example.**

$\vec{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$



$\vec{v} - \vec{u} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$d(\vec{u}, \vec{v}) = \sqrt{\begin{pmatrix} -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \end{pmatrix}} = \sqrt{5}$


p1
p2

7. **Scalar equations of Lines and planes**.

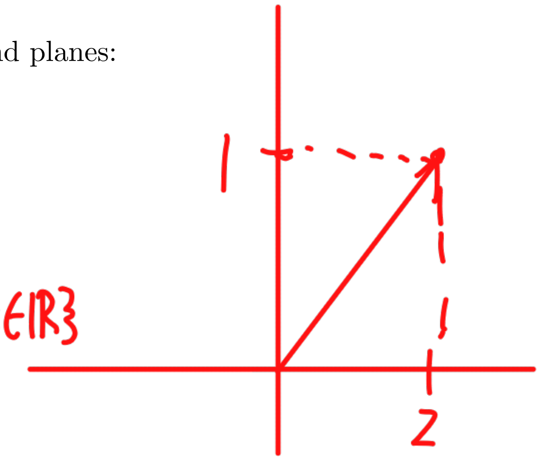
We have seen the vector and parametric description of both lines and planes:

$L_1 = \{t\vec{v}_1 \mid t \in \mathbb{R}\}$

$L_2 = \{v_1 + t\vec{v}_2 \mid t \in \mathbb{R}\}$ $\quad \vec{v}_1 \neq k\vec{v}_2$

$L_1 = \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ $\quad \vec{x} = tv_{11}$, $\vec{y} = tv_{12}$

Plane $P_1 = \{\vec{v}_1 + t\vec{v}_2 + s\vec{v}_3 \mid t,s \in \mathbb{R}\}$




And we know how to represent a line with a $\boxed{\text{scalar equation:}}$

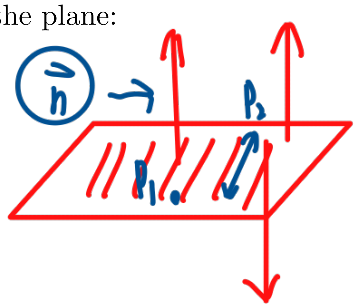
$y = ax$

$y = ax + b$, $b \neq 0$ (not through origin)

$Ax + By = C$

Plane $Ax + By + Cz = D$

We define the **normal** vector to a plane as a vector that is orthogonal to any line segment in the plane:




$\vec{v} = P_2 - P_1$

$\vec{n} \cdot \vec{v} = 0 \quad \vec{n} \cdot (\vec{P}_2 - \vec{P}_1) = 0$

And we use that to find the scalar equation of a plane:

8. A bunch of exercise to do!

(a) Describe a line in $R^2$ passing through point $\mathbf{x_0} = (1, 2)$ and parallel to vector $\mathbf{v} = (3, 4)$.

(b) Do the lines $(x, y, z) = (2, 0, 1)+r(3, -1, 0)$ and $(x, y, z) = (5, 1, -4)+s(1, 0, 2)$ intersect?

(c) What is the vector equation of the line passing through points $\mathbf{x_0}$ and $\mathbf{x_1}$ in $R^3$?

(d) What is the vector and parametric equation of the line passing through points $(0, 1, 1)$ and $(1, 1, 0)$

(e) Describe points of the plane passing through point $\mathbf{x_0}$ which are perpendicular to vector $\mathbf{n}$ in $R^3$

(f) Find the point-normal and general equations of the plane through $(1, 1, 2)$ with normal $(-1, 2, 1)$

(g) Describe the plane in $R^3$ given by the equation $1(x - 3) + 2(y + 2) - 3(z - 1) = 0$

(h) Describe points of a plane passing through a point $\mathbf{x_0}$ and parallel to two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ (that are not parallel) in $R^n$

(i) Find a vector/parametric equation of the plane passing through points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.