3.1 Operations on Matrices

Quote. "By natural selection our mind has adapted itself to the conditions of the external world. It has adopted the geometry most advantageous to the species or, in other words, the most convenient" Jules Henri Poincaré (18541912)

Vocabulary.

- Matrix - a rectangular array of numbers.

1. **Matrices**

   A **matrix** is a rectangular array of numbers (called **entries**):

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A$$

   The size of this matrix is $M \times N$ (don't backwards them up)

   M= # rows
   N= # columns

   To specify the names of entries and the size of the matrix we will sometimes use the notation:

$$A = [a_{ij}]_{m \times n} \qquad = [a_{ij}]$$

   or symbol $(A)_{ij}$ represents the entry of A in row i and column j. Two matrices are equal if they are the same size and all of the entries are the same.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

   (all #s have to be in the same place to be equal)

   $(A)_{21} = 3$ $\qquad A_{22} = 4$

   (2nd row, 1st col)

   A **square matrix** is a matrix with the same number of rows and columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

   $n = m$

   The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** or the **main diagonal**.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A_{3 \times 3}$$

   (diagonal entries are 1,5,9)

2. **Diagonal Matrices**

A square matrix in which all entries off the main diagonal are zero is called a **diagonal matrix**.

Example: $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

3. **Triangular matrices**

- A square matrix in which all entries above the main diagonal are zero is called **lower triangular** (LT), and a square matrix in which all the entries below the main diagonal are zero is called **upper triangular** (UT).
- A matrix that is either upper triangular or lower triangular is called **triangular** (T).

Example: Which one of the following matrices is upper/lower triangular?

(Gauss Elim)
upper
triangular $\rightarrow$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$ $\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ lower triangular

Example: How can we express these properties algebraically?

- A is UT if $a_{ij} = 0$, for every $i > j$.
- A is LT if $a_{ij} = 0$, for every $j > i$.

Example: Can a matrix be both lower and upper triangular?

Yes, diagonal matrices are both.

A^{-1}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A \cdot I = A$
$I \cdot A = A$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Reg World

$6 \cdot 1 = 6$ (Multiplicative
$1 \cdot 6 = 6$ identity element)
$6 \cdot \frac{1}{6} = 1$ (multiplicative inverse)
6^{-1} (does not mean $\frac{1}{x}$)
(same thing to matrix)
(not reciprocal)

4. **Equality of matrices:** Two matrices A and B are equal if they have the same size and their corresponding entries are equal,
that is, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{r\times s}$ then

$$\underbrace{A = B \iff}_{\text{if and only if}} m=r,\ n=s \quad a_{ij}=b_{ij} \quad \forall i \; m \;\&\; j$$

$\forall$ for all

5. **Addition, subtraction and scalar multiplication of matrices.** Addition, subtraction and scalar multiplication:
If matrices A and B have the same size and c is a scalar then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} \quad (A-B)_{ij} = (A)_{ij} - (B)_{ij} \quad (cA)_{ij} = c(A)_{ij}$$

If they **do not** have the same size $A+B$ and $A-B$ are *undefined.*

Example. Given matrices

$$A = \begin{bmatrix} 2 & 0 \\ 7 & -5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 5 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \end{bmatrix}, D = \begin{bmatrix} 6 & 3 \\ 4 & 5 \\ 9 & 8 \end{bmatrix}$$

Evaluate $A+B$, $A+D$, $B-D$, $C-B$, $5A$ and $(-2)B$

$A+B=$ undefined

$$A+D = \begin{bmatrix} 2 & 0 \\ 7 & -5 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 4 & 5 \\ 9 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 11 & 0 \\ 13 & 11 \end{bmatrix}$$

$B-D=$ undefined

$$C-B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \end{bmatrix} \cancel{+} - \begin{bmatrix} 4 & -3 & 0 \\ 1 & 5 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 3 \\ -1 & -6 & 1 \end{bmatrix}$$

$$^{*}5A = 5\begin{bmatrix} 2 & 0 \\ 7 & -5 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 35 & -25 \\ 20 & 15 \end{bmatrix}$$

6. **List of row and column vectors**

Any matrix can be thought of as

(a) a list (row) of columns vectors; or

(b) a list (column) of row vectors.

In general, if $A = [a_{ij}]_{m\times n}$, we can write

$$A = \begin{bmatrix} \mathbf{c_1}(A) & \mathbf{c_2}(A) & \dots & \mathbf{c_n}(A) \end{bmatrix} = \begin{bmatrix} \mathbf{r_1}(A) \\ \mathbf{r_2}(A) \\ \vdots \\ \mathbf{r_m}(A) \end{bmatrix}$$

where $\mathbf{r_i}(A)$ is the i-th row of A and $\mathbf{c_j}(A)$ is the j-th column of A.

Example. What are the column and row vectors of $A = \begin{bmatrix} 3 & 0 & 7 & 5 \\ 2 & 9 & 6 & 0 \\ 1 & 8 & 1 & 4 \end{bmatrix}$

$A = \left[\, \vec{c_1} \mid \vec{c_2} \mid \vec{c_3} \mid \vec{c_4} \,\right]$ $\quad \vec{c_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\quad \vec{c_2} = \begin{bmatrix} 0 \\ 9 \\ 8 \end{bmatrix}$

$A = \begin{bmatrix} \vec{r_1} \\ \hline \vec{r_2} \\ \hline \vec{r_3} \end{bmatrix}$ $\quad \vec{r_1} = \begin{bmatrix} 3 & 0 & 7 & 5 \end{bmatrix}$ $\quad$ >>A=[3 0 7 5; 2 9 6 0; 1 8 1 4]

$\vec{r_2} = \begin{bmatrix} 2 & 9 & 6 & 0 \end{bmatrix}$

$\vec{r_3} = \begin{bmatrix} 1 & 8 & 1 & 4 \end{bmatrix}$

7. **Span and linear independence for Matrices**

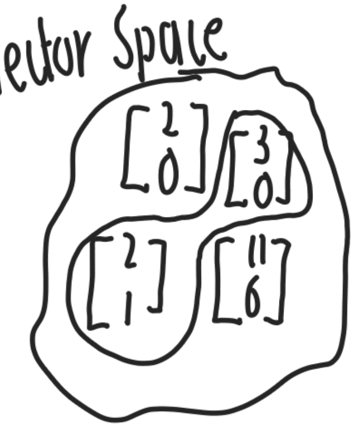
These follow directly from the same definitions we had for vectors.

Let $\mathscr{B} = \{A_1, ..., A_k\}$ be a set of matrices.

The **span** of $\mathscr{B}$ is defined as Span $\mathscr{B} = \{t_1 A_1 + ... + t_k A_k\}$

The set $\mathscr{B}$ is linearly independent if the only solution to $c_1 A_1 + ... + c_k A_k = 0$ is $c_1 = ... = c_k = 0$.

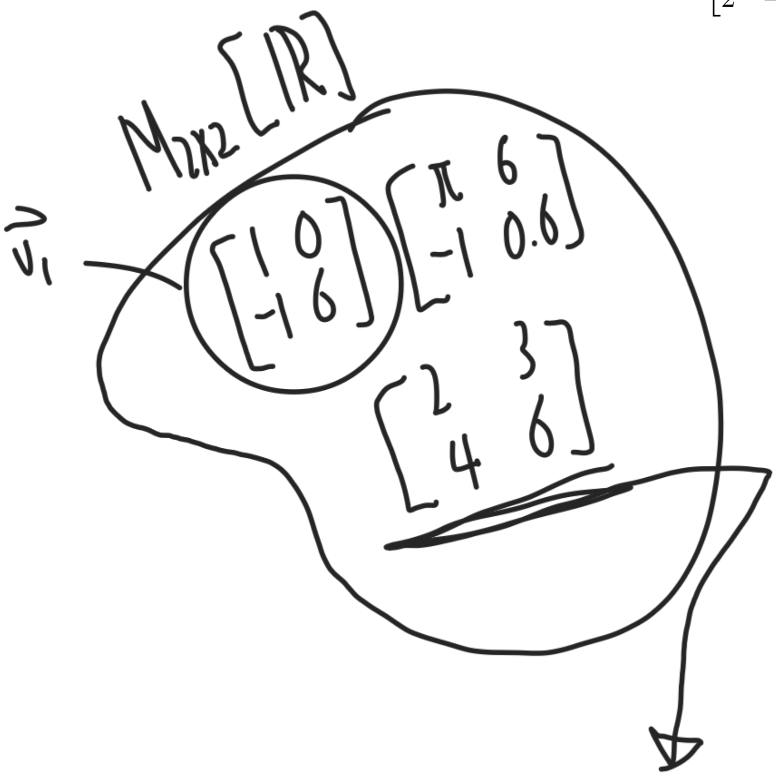
In the textbook

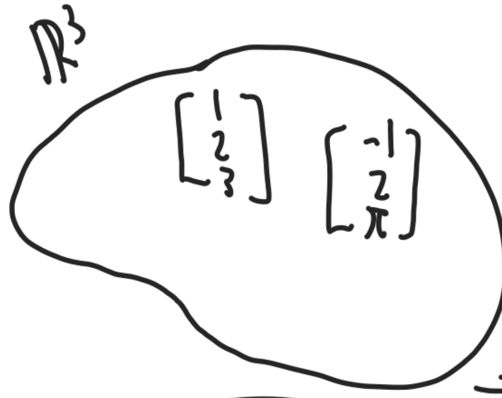


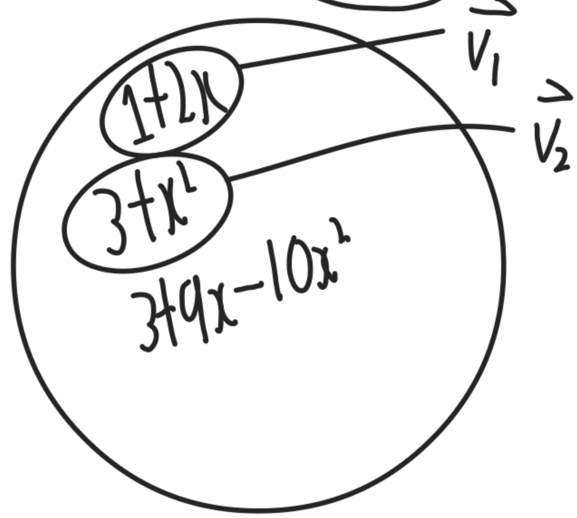
8. **Example**

Let $\mathscr{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \right\}$

and determine if the set is linearly independent. Is $\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}$ in the Span $\mathscr{B}$?







Basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

9. **Transpose**

If A is an $m \times n$ matrix, then the **transpose** of A, denoted as A^T, is the $n \times m$ matrix created by making the rows of A into columns. We can write this as $(A^T)_{ij} = (A)_{ji}$.

>>A

>>A'

Example. Suppose $A = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 7 & 1 \end{bmatrix}$. Find A^T, B^T and $(B^T)^T$.

$$A^T = \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \quad B^T = \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix} \quad (B^T)^T = B$$

10. **Matrix vector multiplication** ((Cannot stress how important this is))

Example. Consider the following linear system

$$\boxed{\begin{aligned} 3x_1 - 7x_2 + x_3 &= 4 \\ 5x_1 + x_2 + 2x_3 &= 2 \end{aligned}}$$

We can naturally associate three matrices with the system:

$$A = \begin{bmatrix} 3 & -7 & 1 \\ 5 & 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

A is usually called the **coefficient matrix** of the system.

We want to define the *product* $A\mathbf{x}$ in such a way that the $m = 2$ linear equations can be rewritten in a single matrix equation $A\mathbf{x} = b$.

$A\vec{x} = \vec{b}$

Note that $m = 2$ linear equations can be written as a single matrix equation as follows:

$$\begin{bmatrix} 3 & -7 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 3 & -7 & 1 & 4 \\ 5 & 1 & 2 & 2 \end{array}\right]$$

$$\boxed{A \cdot \vec{x} = \vec{b}}$$

(Dot Product)

$3x_1 - 7x_2 + x_3 = 4$

$5x_1 - x_2 + 2x_3 = 2$

The product $A\mathbf{x}$ (definition):

Suppose A is an $m \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ column vector. The **product** $A\mathbf{x}$ is defined as:

$$A\mathbf{x} = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ a_{21} \quad a_{22} \quad \cdots \quad a_{2n} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Remark: The result of multiplication is $m \times 1$ column vector. The product is only defined when the number of columns of A is the same as the number of rows of $\mathbf{x}$.

Example:

$$\underbrace{\begin{bmatrix} \boxed{1 \quad 2 \quad -1} \\ 0 \quad 3 \quad 2 \\ 4 \quad 5 \quad 3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 13 \end{bmatrix}}_{3\times 1}$$

$(1)(2)+(2)(1)+(-1)(0)$

$(0)(2)+(3)(1)+(2)(0)$

(3 row, 1 col) (Don't get it backwards)

We can also view the product $A\mathbf{x}$ as a linear combination of column vectors of A:

$x_1 + 2x_2 + 3x_3$

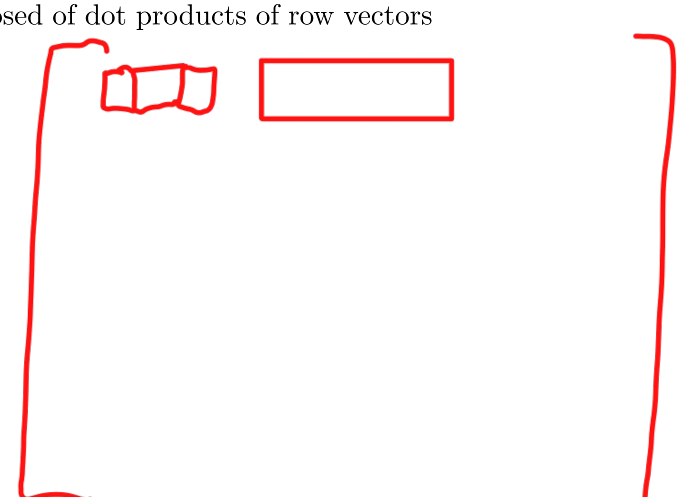
$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\mathbf{c_1}(A) + x_2\mathbf{c_2}(A) + \cdots + x_n\mathbf{c_n}(A)$$

Ex.

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \\ 7 \quad 8 \quad 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Or we can view the product $A\mathbf{x}$ as a column vector composed of dot products of row vectors of A and $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{r_1}(A)} \cdot \mathbf{x} \\ \mathbf{r_2}(A) \cdot \mathbf{x} \\ \cdots \\ \mathbf{r_m}(A) \cdot \mathbf{x} \end{bmatrix}$$



(Screen as Matrix of Pixels)

Example. Evaluate the following using all three methods: $(5)(3)+(0)(-2)+(2)(10)=35$

$$\begin{bmatrix} 5 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 35 \\ -1 \end{bmatrix}$$

$(2\times\boxed{3)(3}\times 1)$

Mid #s match $= (2\times 1)$

11. **The product** AB

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. The **product** AB is the $m \times p$ matrix defined as

$$\begin{bmatrix} A\mathbf{c_1}(B) & A\mathbf{c_2}(B) & \dots & A\mathbf{c_p}(B) \end{bmatrix}$$

Example. Given $A = \begin{bmatrix} 4 & 0 & 5 \\ 3 & -2 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 10 & -2 & 0 \\ 0 & 6 & 1 & 5 \\ 2 & 2 & -7 & 9 \end{bmatrix}$, find AB and BA.

$$AB = \begin{bmatrix} 4 & 0 & 5 \\ 3 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 10 & -2 & 0 \\ 0 & 6 & 1 & 5 \\ 2 & 2 & -7 & 9 \end{bmatrix} =$$

$(2\times\boxed{3)\ (3}\times 4)$ 2×4

$$= \begin{bmatrix} 14 & 50 & -43 & 45 \\ 19 & 34 & -64 & 62 \end{bmatrix}$$

$(3)(-2)+(-2)(1)+(8)(-7)$

$$BA = \begin{bmatrix} 1 & 10 & -2 & 0 \\ 0 & 6 & 1 & 5 \\ 2 & 2 & -7 & 9 \end{bmatrix} \begin{bmatrix} 4 & 0 & 5 \\ 3 & -2 & 8 \end{bmatrix}$$ No!

3×4 2×3

$AB \neq BA$

Not cummutative

$$2 \cdot 3 = 3 \cdot 2$$

$$AB \neq BA$$

$$\cancel{X} \cdot AB = A\cancel{X}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}} = \underset{AB}{\begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}}$$

$$\underset{B}{\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \underset{BA}{\begin{bmatrix} -2 & -2 \\ 9 & 12 \end{bmatrix}}$$

12. **Linearity properties** If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then

(a) $A(c\mathbf{u}) = c(A\mathbf{u})$ $\leftarrow$ $A(c\vec{u}) = c(A(\vec{u})$

(b) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

• Matrices are...

13. **IMPORTANT properties of matrix mutliplication**

- Matrix multiplication is NOT commutative. $AB \neq BA$
- The cancellation law is NOT valid.
- $AB = 0$ does not mean that one of A or B is zero.

$\cancel{A}B = \cancel{A}C$ Not valid!

$AB = 0$ does not mean $A = 0$ or $B = 0$

$xy = 0$

It does NOT mean that $B = C$

14. **The Identity Matrix**

$I_{2 \times 2} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

plays the role of 1.

$A \cdot I = A$

$I \cdot A = A$

• exception

• multiplicative identity element