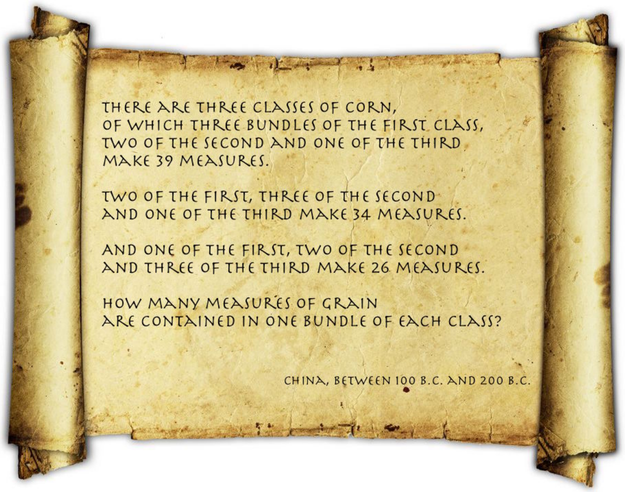
2.1 Systems of Linear Equations and Elimination



Vocabulary.

- linear equation: equations of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Geometrically a hyperplane in $\mathbb{R}^2$.

  i.e: $2x_1 + 3x_2 = \boxed{0}$ Homogeneous linear eqn

- homogeneous linear equation: b is zero.
- non-linear equation: all the rest

  $x+y=3$
  $-x+2y=6$

- system of equations: a set of equations in the same variables.
- unknowns: the variables in a system of equations (what we are trying to find)
- solution: the values of the variables that solve the equation(s).
- solution set: the set of all the solutions.
- consistent: a system of linear equations that has at least one solution.
- inconsistent: a system of linear equations that has no solution.

  $x+y=0$
  $x+y=10$
  Inconsistent

- pivot: top non-zero entry in a column of a matrix. ← we're gonna see

1. **Definition of a Linear equation**

   A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

   $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

   where $a_1, a_2, \ldots, a_n$ and b are constants such that not all a_i's equal 0.

   If $b = 0$, then the equation is called a **homogeneous linear equation**.

2. **Examples**

   → $2x_1 + 3x_2 = 0$ (it is homogen.)

   → $2x_1 + 3x_2 = -6$ (not homogen.)

   $3x_1 + 4x_2 + 5x_3 - x_4 = 0$ ←

   $x_1^{2} + x_2 = 0$
   Not linear

   $\ln(x_1)$, $\sin(x_2)$ > Not linear

   $x_1x_2 + x_3 = 0$
   No! No product!

   $\sqrt{x_1} + x_2 = 0$ Not linear

3. **Question.** What does a linear equation in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ look like (geometrically)?

Line, Planes, Hyperplanes

4. **Definition of a System of linear equations**

A finite set of linear equations is called a **system of linear equations** or a **linear system**. The variables in such a system are called **unknowns**.

In general, a linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be expressed as

$$\begin{array}{ccccccc} a_{11}x_1+ & a_{12}x_2+ & \ldots+ & a_{1n}x_n= & b_1 \\ a_{21}x_1+ & a_{22}x_2+ & \ldots+ & a_{2n}x_n= & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & \ldots+ & a_{mn}x_n= & b_m \end{array}$$ ← homogenous if $b=0$

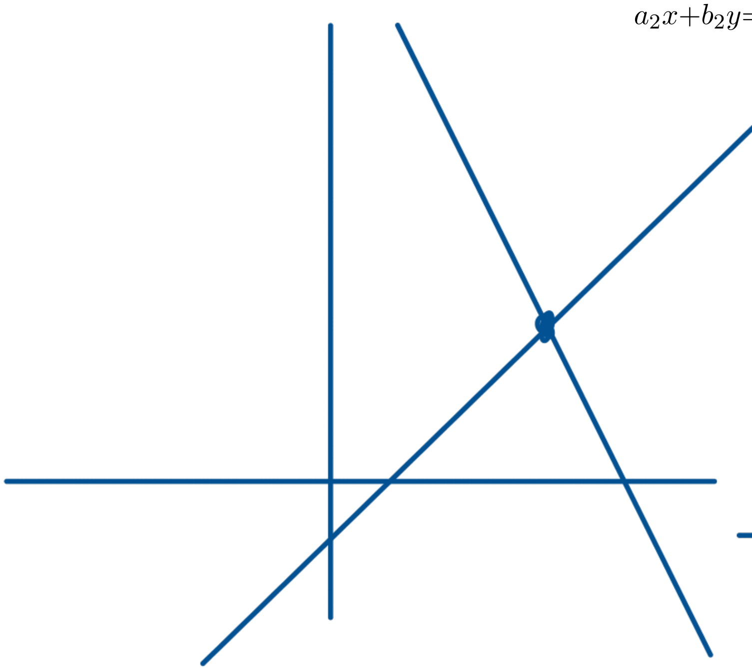
5. **Example**

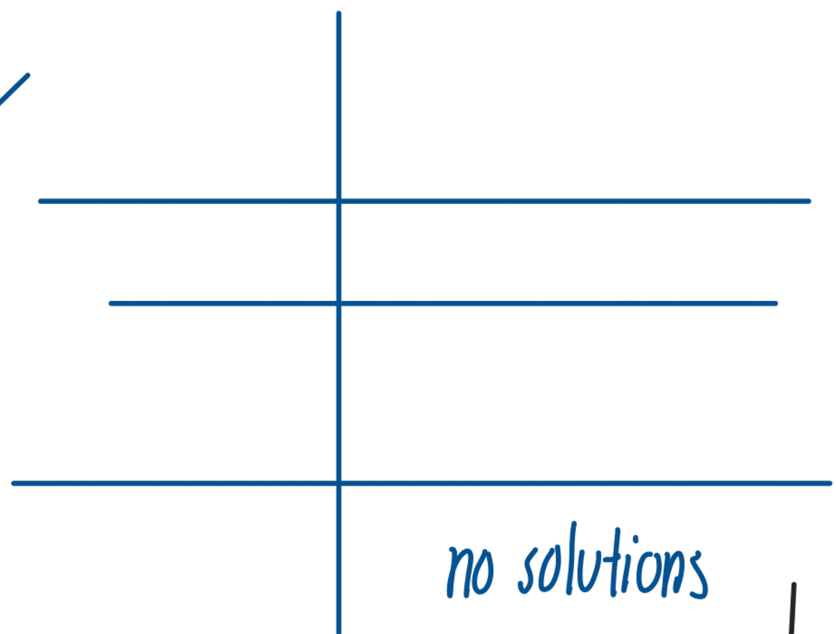
$2x_1+3x_2=5$
$4x_1-2x_2=8$

(x_1, x_2)

6. **Example.** Consider a linear system of 2 equations in 2 unknowns x and y. What are the possible solution sets of this system?

$$\left.\begin{array}{l} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{array}\right\}$$ one solution





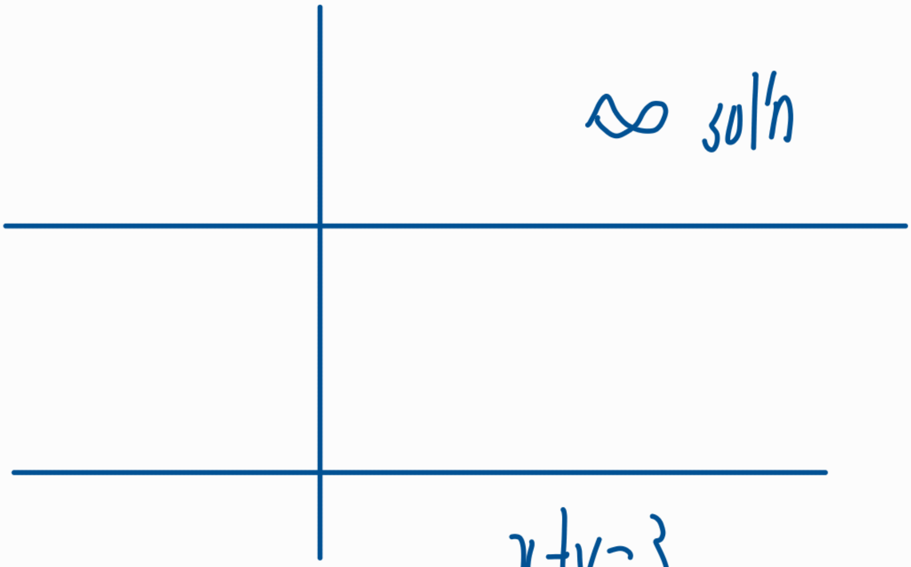
no solutions

7. **Definition of a Solution Set**

A **solution** to a linear system is a sequence of n numbers $s_1, s_2, \ldots, s_n$ that when substituted for $x_1, x_2, \ldots, x_n$, respectively, satisfy all the equations. We can represent each solution as a point/vector in n-space: $(s_1, s_2, \ldots, s_n)$.

The set of all solutions is called the **solution set**.

• can have none, 1 element or ∞



$x+y=3$

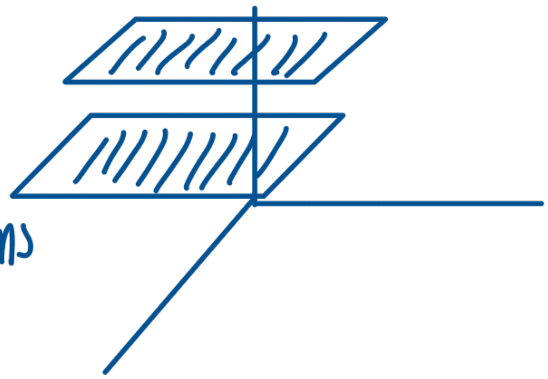
$2x+2y=6$

- 0, 1, or infinite solutions

8. **Example.** Consider a linear system of 2 equations in 3 unknowns x, y and z. What are the possible solution sets of this system?

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$

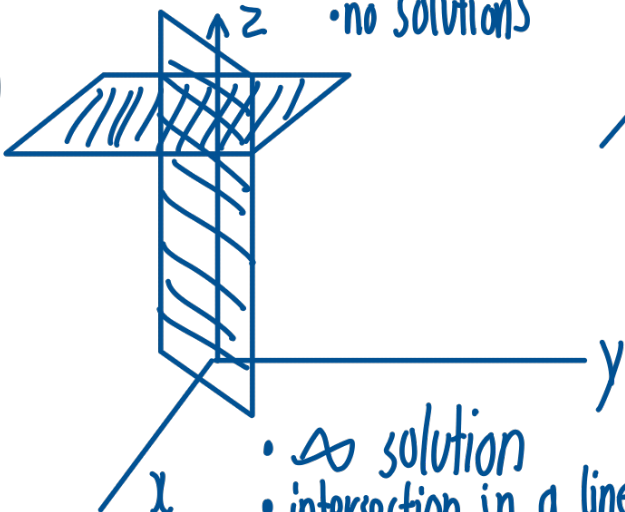
① Same plane – ∞ member of solutions



② Parallel planes

• no solutions

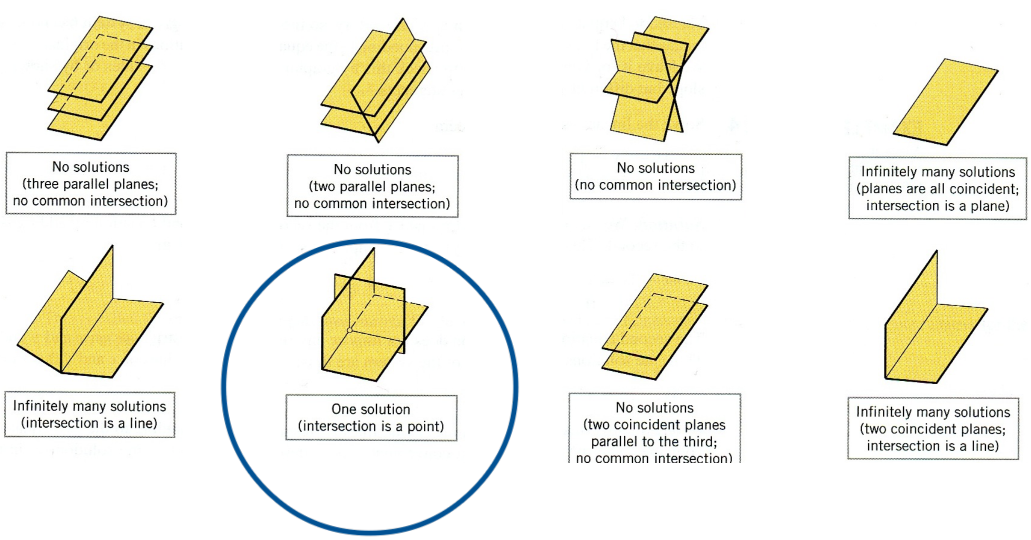
③



• ∞ solution

• intersection in a line

Consider a linear system of 3 equations in 3 unknowns x, y and z. What are the possible solution sets of this system?



Theorem 0.1 *Every linear system has either zero, one, or infinitely many solutions.*

0

1

∞

$x+y=5$

$x+y=10$

If a linear system has at least one solution it's called **consistent**, otherwise it's called **inconsistent**.

Given any linear system, we can use the following operations to make a new system that has *exactly* the same solutions:

(a) Multiply one of the equations by a non-zero scalar;

(b) Interchange two equations;

(c) Add a multiple of one equation to another.

9. **Example**

① $2x_1 - x_2 = 4$ has sol'n $(7,10)$

② $-2x_1 + 2x_2 = 6$

$x_2 = 10$

a) ① $x_1 - \frac{1}{2}x_2 = 2$ $\quad 7-5=2$ ✓

② $-2x_1 + 2x_2 = 6$ $\quad -14+20=6$ ✓

b) ① $-2x_1 + 2x_2 = 6$ all is well!

② $2x_1 - x_2 = 4$

c) $2(2x_1 - x_2 = 4) + (-2x_1 + 2x_2 = 6)$

① $2x_1 + 0x_2 = 14$

② $-2x_1 + 2x_2 = 6$

10. **Augmented matrix**

Given a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

we can abbreviate the system by ignoring variables and = symbols using the matrix notation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right].$$

• rectangular array of #s

This matrix is called the **augmented matrix for the linear system.**

11. **Example.** What is the augmented matrix of the following system?

$$\begin{aligned} x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \\ 2x_1 + 3x_2 - x_3 + 5x_4 &= -5 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -3 & 4 \\ 2 & 3 & -1 & 5 & -5 \end{array}\right]$$

$x_1 + 2x_2 = 3$
$6x_1 + 9x_2 = -4$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 6 & 9 & -4 \end{array}\right]$$

Modifications of the linear system correspond to **elementary row operations** on the corresponding augmented matrix:

(a) Multiply one of the equations by a non-zero scalar;
(b) Interchange two equations;
(c) Add a multiple of one equation to another.

same

(a) Multiply a row by a non-zero scalar;
(b) Interchange two rows;
(c) Add a multiple of one row to another.

12. **Example.** Write the following linear system as an augmented matrix, simplify the matrix using row operations, and finally solve the system

$$\begin{aligned} 2x - 3y &= 1 \\ x - y &= 3. \end{aligned}$$

$2(8) - 3(5) = 1$ ✓
$8 - 5 = 3$ ✓

$$\left[\begin{array}{cc|c} 2 & -3 & 1 \\ \boxed{1} & -1 & 3 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 2 & -2 & 6 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\left[\begin{array}{cc|c} 2 & \boxed{-3} & 1 \\ 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cc|c} 2 & 0 & 16 \\ 0 & 1 & 5 \end{array}\right] \quad R_1 \to \tfrac{1}{2}R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 5 \end{array}\right] \begin{array}{l} \leftarrow x_1 + 0x_2 = 8 \\ \leftarrow 0x_1 + x_2 = 5 \end{array}$$

$$x_1 = 8$$

$$x_2 = 5$$

13. **Definition of (Reduced) row echelon forms**

A matrix is said to be in **row echelon form** if it has the following properties:

(a) Each row either consists of all zeros, or has a 1 as its first nonzero entry. (This entry is called a **leading 1**.)

(b) Any rows of all zeros are grouped together at the bottom of the matrix.

(c) In two successive nonzero rows, the *leading 1* in the *lower* row is to the *right* of the *leading 1* in the *upper* row.

14. **Examples** Which of the following matrices are in row echelon form? Circle *leading 1's*. These positions in the matrix are called **pivot positions**.

REF= row echelon form

RREF= reduced "

No

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

•violates b)

Yes, REF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

No!

$$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• To the left?

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 1 & 0 & 1 & 5 \end{bmatrix}$$

No

Not in RREF

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

REF (0s at bottom, 1 to right)

RREF

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

REF

Not RREF

No!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\neq 1$

• row operations to REF (sometimes RREF)

Matlab rref

↳ returns reduced echelon from matrix

15. **Solving linear systems in "reduced RE form"**

Solve the linear system with variables x, y, z that has the augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & \boxed{0} & \boxed{1} \end{array}\right]$

$x_1 + 0x_2 + 0x_3 = 5$

$0x_1 + x_2 + 0x_3 = -2$

$0x_1 + 0x_2 + 0x_3 = 1$

$0 \neq 1$

· inconsistent

· no solution

Solve linear system with variables x_1, x_2, x_3, x_4 that has the augmented matrix $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -3 \\ 0 & 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

RREF

$x_1 + 2x_2 + 0x_3 + 2x_4 = 3$

x_1 & x_3 are leading variables

x_2 & x_4 are free variables

Let $x_4 = t$

$x_2 = s$

$x_1 + 2x_2 + 2x_4 = -3$

$x_1 = -3 - 2s - 2t$

$x_3 - 5x_4 = 8$

$x_3 = 8 + 5x_4$

$x_3 = 8 + 5t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 8 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$-3 - \underline{\underline{2s}} - \underline{2t} = x_1$$

- Plane not through the origin

16. **Solving linear systems in "RE form" — Back substitution**

Solve the linear system with variables x_1, x_2, x_3, x_4 that has the augmented matrix $\begin{bmatrix} 1 & \frac{2}{7} & \frac{3}{7} & \frac{2}{7} & 3 \\ 0 & 1 & -2 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

- **Step 1.** Express the leading variables.
- **Step 2.** Beginning with the bottom equation and working *upward*, substitute each equation into all equations above it.
- **Step 3.** Assign different parameters to all free variables, if any.

17. **Gaussian elimination**

Purpose: Converts an augmented matrix to *row echelon form* (**introduce 0's below leading 1's**).

Convert the following augmented matrix to *row echelon form*:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 18 & 9 & -6 \\ 4 & 8 & 12 & 10 & 4 \end{bmatrix}$$

- **Step 1.** Locate the leftmost column c that does not consist entirely of zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 18 & 9 & -6 \\ 4 & 8 & 12 & 10 & 4 \end{bmatrix}$$

- **Step 2.** Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column c.

- **Step 3.** If the entry in the top row and column c is a, multiply the top row by $1/a$ (in order to introduce a leading 1).

- **Step 4.** Add suitable multiples of the top row to the rows below so that all entries below leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 9 & -3 & -9 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

second row:
third row:
fourth row:

- **Step 5.** Cover the top row, and if there are any nonzero rows left, repeat Step 1.