1.5 Dot Products and Projections in $\mathbb{R}^n$

Quote. "The important thing to remember about mathematics is not to be frightened" Richard Dawkins (1941-)

Vocabulary.

- Scalar Product: another name for the dot product (the result of the dot product is a scalar)
- Inner product: another name for the dot product.
- Projection: an everyday example of this is your shadow - a projection of you onto the ground.
- Unit Vector: a vector with a norm of 1
- Dimension of a space: the number of vectors in the basis that spans the space.
- Hyperplane: n-dimensional generalization of the plane. A subspace whose dimension is one less than the ambient space.

Here generalize the dot product from $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$ and we introduce the important idea of projection.

1. **Dot Products, Orthogonality, and Norms in $\mathbb{R}^n$**

   By way of series of examples, we will see how these operations generalize to $\mathbb{R}^n$

   $(1,2)\cdot(-1,5) = (1)(-1)+(2)(5) = 9$

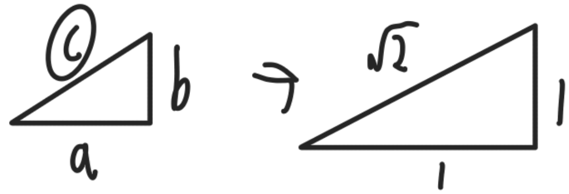
   $$(1,2,3,4)\cdot(-1,0,0,1) = (1)(-1)+(2)(0)+(3)(0)+(4)(1)$$
   $$= 3$$

   in any $\mathbb{R}^n$ $\vec{u}\cdot\vec{v}=0$ $\vec{u}\perp\vec{v}$ (orthogonal)

2. **The Triangle Inequality**

   $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|$

   $\boxed{\text{c is less than a+b}}$ in $\mathbb{R}^2$



$\sqrt{2}$ less than those 2

3. **The Cauchy-Schwartz Inequality** (complicated to prove, not intuitive)

$$|\vec{u}\cdot\vec{v}| \leq ||\vec{u}||\,||\vec{v}||$$

Fall 2020 video has a proof.

4. **Hyperplane**

Recall the method of finding the scalar equation of a plane in $\mathbb{R}^3$:

- Test points $\vec{x}$ and $\vec{p}$, $\vec{p}$ fixed, $\vec{x}$ any other point

$\vec{n}\cdot(\vec{x}-\vec{p})=0$ (normal line)

$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

$n_1(x_1-p_1)+n_2(x_2-p_2)+n_3(x_3-p_3)=0$, $n_1x_1+n_2x_2+n_3x_3 = \vec{n}\cdot\vec{p}$

Using the same method we can find scalar equation of the Hyperplane in $\mathbb{R}^n$

$\vec{m}$ orthogonal to $\vec{x}-\vec{p}$ $\quad \vec{m}, \vec{x}, \vec{p} \in \mathbb{R}^n$

$m_1x_1+m_2x_2+\dots+m_nx_n=\vec{m}\cdot\vec{p}$

in the scalar eq'n of the hyperplane.

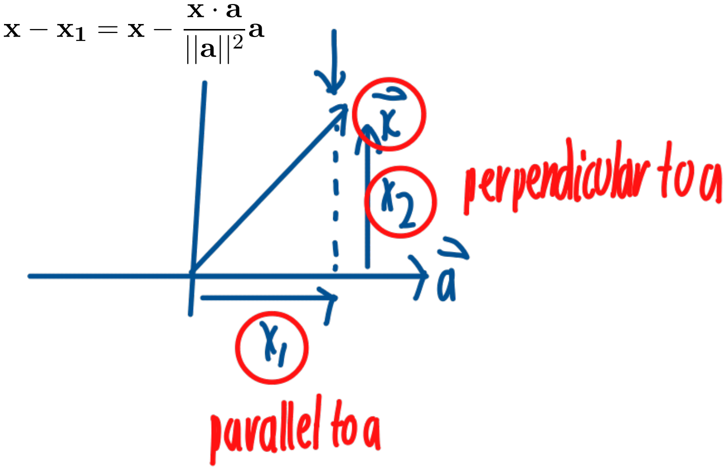
5. **Orthogonal projections onto lines in $\mathbb{R}^n$**

Theorem 0.1 *If* $\mathbf{a}$ *is a non-zero vector in* $\mathbb{R}^n$*, then every vector* $\mathbf{x}$ *in* $\mathbb{R}^n$ *can be expressed in exactly one way as*

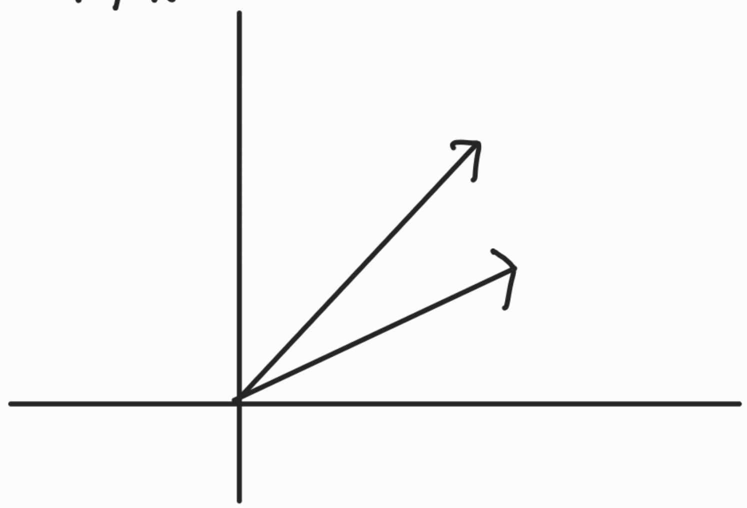
$$\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$$

where $\mathbf{x_1}$ *is in* $\operatorname{span}\{\mathbf{a}\}$ *and* $\mathbf{x_2}$ *is perpendicular to* $\mathbf{a}$*. The vectors* $\mathbf{x_1}$ *and* $\mathbf{x_2}$ *are given by*

$$\mathbf{x_1} = \underbrace{\frac{\mathbf{x}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}}_{\text{projection of } \vec{x} \text{ onto } \vec{a}} \quad \textit{and} \quad \mathbf{x_2} = \mathbf{x} - \mathbf{x_1} = \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$



Disregard...



Definition: **orthogonal projection onto line in** $\mathbb{R}^n$

If $\mathbf{a}$ is a non-zero vector in $\mathbb{R}^n$ and $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then the **orthogonal projection of x onto the line** span$\{\mathbf{a}\}$ is denoted by $\text{proj}_{\mathbf{a}}\mathbf{x}$ and defined by

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

The vector $\text{proj}_{\mathbf{a}}\mathbf{x}$ is also called the **vector component of x along a** and $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}$ is called the **vector component of x orthogonal to a**.

Note: the portion of the vector that is perpendicular to the projection is denoted $\text{perp}_{\mathbf{a}}\mathbf{x}$.

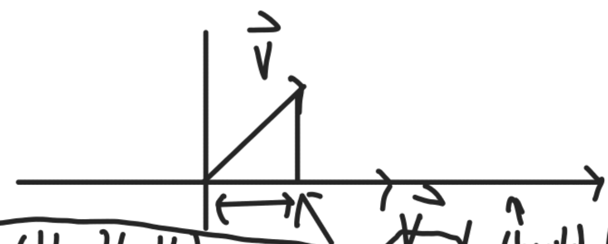
Example.

$\vec v = (1/3, -2/3, 2/3)$ $\vec u = (4,1,-3)$

Projection of $\vec u$ into $\vec v$
Don't get direction wrong!
Know what you're trying to calculate

$\text{proj}_{\vec v}(\vec u)$ and $\text{perp}_{\vec v}(\vec u)$

$$\text{proj}_{\vec v}(\vec u) = \frac{\vec u \cdot \vec v}{\|\vec v\|^2} \cdot \vec v$$



$$= \frac{(4,1,-3)\cdot(1/3,-2/3,2/3)}{(1/3)^2+(-2/3)^2+(2/3)^2} \cdot (1/3,-2/3,1/3) = (-4/9, 8/9, -8/9)$$

$\vec u - \text{proj}_{\vec v}(\vec u) = \text{perp}_{\vec v}(\vec u)$

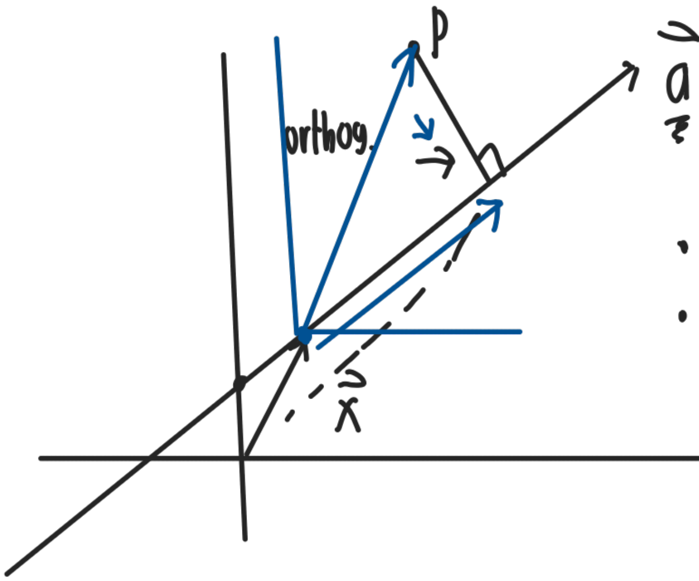
• Should length affect? NO!
• remove dependency of it!

6. **The ideas of minimum distance, nearest point, and best approximation.**

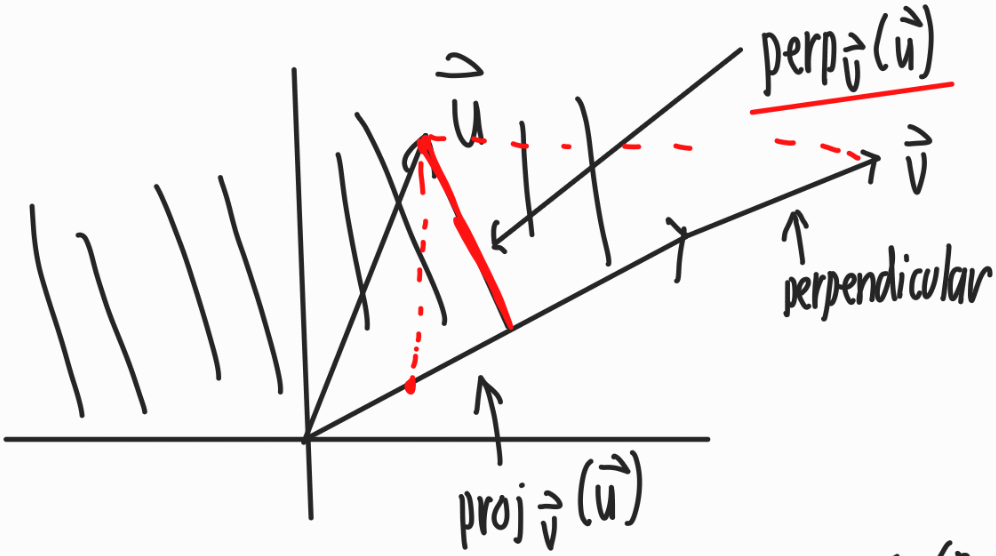
$(4,1,-3) - (-4/9, 8/9, -8/9)$
$= (40/9, 1/9, -19/9)$

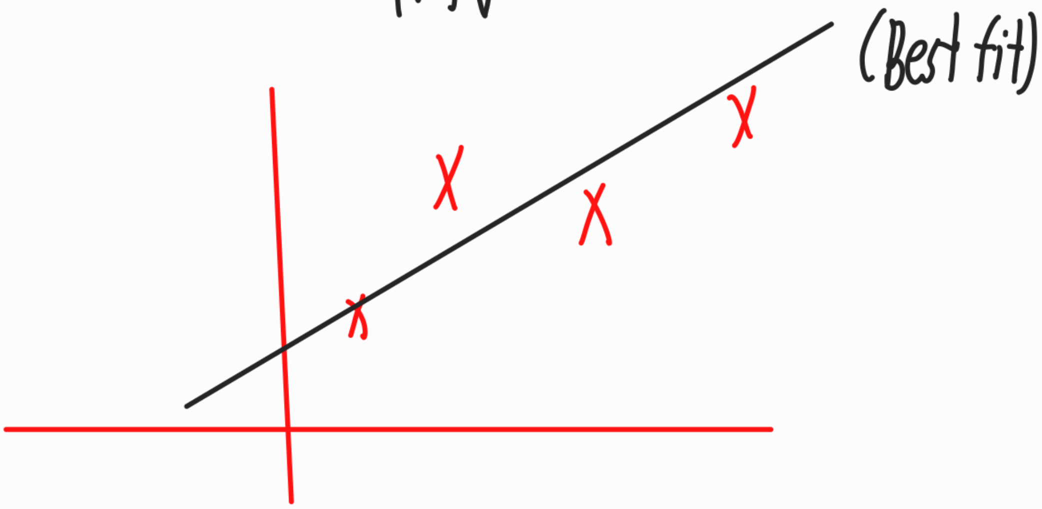
• Be prepared of fractions on midterm or final (More below)

least squares.



• Min dist from P to $\vec a$
• complication
• Is line $\vec a$ a subspace? No!
• $\vec x$ DNE when put in origin?


$\vec{u}$
$\text{perp}_{\vec{v}}(\vec{u})$
$\vec{v}$
perpendicular
$\text{proj}_{\vec{v}}(\vec{u})$


(Best fit)

$\checkmark$

7. **A bunch of exercises to do!**

(a) Find the orthogonal projection of $\mathbf{x}$ onto the span$\{(2,3)\}$. Draw the graph.

(b) Find the orthogonal projection of $\mathbf{x}$ onto the the line $x + 3y = 0$. Draw the graph.

(c) Given $\mathbf{x} = (7, -5, 9, -1)$ and $\mathbf{a} = (3, 0, 1, 2)$, find the vector components of $\mathbf{x}$ along $\mathbf{a}$ and orthogonal to $\mathbf{a}$.

Quiz 2 - 1.3
- 1.5 from assignment 2
Not 2.1