1.4 Vectors in $\mathbb{R}^n$

Quote. "I have photographed many people: artists, writers, and scientists, among others. In speaking about their work, mathematicians use the words 'elegance', 'truth', and 'beauty' more than everyone else combined." Mariana Cook, American photographer (1955-)

Vocabulary.

- Closed under addition: if addition of two members of a set always produces another member of the set.
- Closed under scalar multiplication: if multiplying a member of a set by a scalar always produces another member of the set.
- $\mathbb{R}^n$: A space like $\mathbb{R}^2$ or $\mathbb{R}^3$ but the vectors have n components.
- Subspace: a subset of a space, closed under addition and scalar multiplication.

In this section we generalize what we have done in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$ and we introduce the important idea of a subspace. In generalizing to $\mathbb{R}^n$, we will repeat some of what we have said before, with small extensions. This also serves as a review of the topics we have covered so far.

1. **Equality, addition, scalar multiplication of vectors in $\mathbb{R}^n$**

   By way of series of examples, we will see how these operations generalize to $\mathbb{R}^n$

2. **Closure a under given operation**

Definition: Closed under addition

A non-empty subset S of $\mathbb{R}^n$ is **closed under addition** if for all $\mathbf{u}$ and $\mathbf{v}$ in S, $\mathbf{u}+\mathbf{v}$ is also in S.

Let $S_1 = \{(1,0,0),(0,1,0)\}$. Is S_1 closed under addition?

Let $S_2 = \{(a,b,0);\ a,b \in \mathbb{Z}\}$. Is S_2 closed under addition?

Definition: Closed under scalar multiplication

A non-empty subset S of $\mathbb{R}^n$ is **closed under scalar multiplication** if for all $\mathbf{u}$ in S and all c in $\mathbb{R}$, $c\mathbf{u}$ is also in S.

Is S_2 closed under scalar multiplication?

Let $S_3 = \{(x,y,0);\ x,y \in \mathbb{R}\}$. Is S_3 closed under scalar multiplication?

3. **Subspaces of $\mathbb{R}^n$**

Definition: Subspace

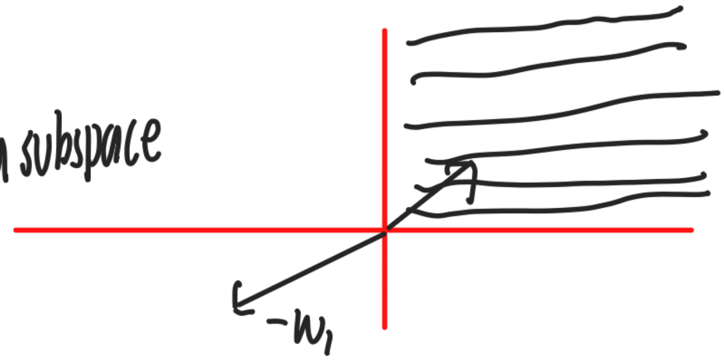
A **subspace** of $\mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$ that is closed under *addition* and under *scalar multiplication.*

Is the set $W = \{(x, y);\ x > 0, y > 0\}$ a subspace of $\mathbb{R}^2$?

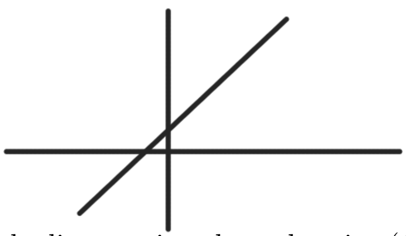
No, $\vec{w}_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$(-1)\vec{w}_1 = \begin{pmatrix}-1\\-1\end{pmatrix}$

Not a subspace



Is a line in $\mathbb{R}^n$ passing through origin a subspace of $\mathbb{R}^n$?



Yes

Every subspace contains no zero vector

Is the line passing through point $(1, 0, 0)$ and parallel with vector $(0, 1, 1)$ a subspace of $\mathbb{R}^3$?

$W = \{\vec{x} : \vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}\}$

No

$\vec{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$

$\vec{x}_2 = \begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\1\end{pmatrix}$

$\vec{x}_1 + \vec{x}_2 = \begin{pmatrix}2\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\1\end{pmatrix}$

Is a plane in $\mathbb{R}^n$ passing through origin a subspace of $\mathbb{R}^n$?

Yes

Is the set $\{\mathbf{0}\}$ a subspace of $\mathbb{R}^n$?

Is the set $\mathbb{R}^n$ a subspace of $\mathbb{R}^n$?

Note: The *zero* subspace and $\mathbb{R}^n$ are called **trivial subspaces** of $\mathbb{R}^n$.

4. **The subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.**

In $\mathbb{R}^2$, the possible subspaces are: the trivial subspace, lines through the origin and all of $\mathbb{R}^2$.

In $\mathbb{R}^3$, the possible subspaces are: the trivial subspace, lines through the origin, planes through the origin and all of $\mathbb{R}^3$.

5. **Spanning sets, linear independence, standard basis in $\mathbb{R}^n$**

By way of series of examples, we will see how these concepts generalize to $\mathbb{R}^n$

Standard Basis $\mathbb{R}^2$ $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

• basis vectors are linearly independent

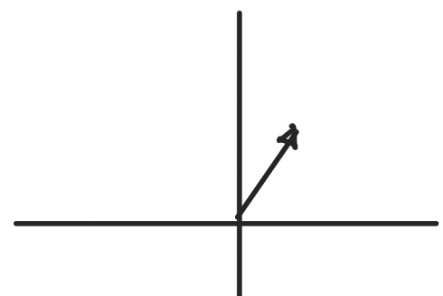
$\mathbb{R}^3$ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$\mathbb{R}^4$ $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$

$\mathbb{R}^n \rightarrow$ n-vectors
n-dimension

$\boxed{\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}} = \mathbb{R}^2$

span(anything) in a subspace



$\begin{bmatrix}1\\1\end{bmatrix}$

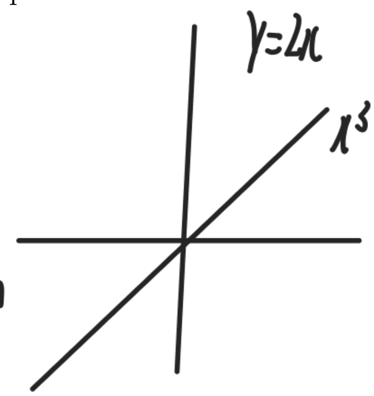
6. **Bases of Subspaces**.

A set of vectors that spans a subspace AND in which all of the vectors are linearly independent is a basis for the subspace.

Example.

Show $B = \left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ is a basis for the hyperplane P in $\mathbb{R}^4$ when scalar eq'n

$x_1 + x_2 + x_3 - x_4 = 0$



$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ Every $\vec{x} \in$ P(lane) satisfies

$x_4 = x_1 + x_2 + x_3$

Pick $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ } x_4 must be 6

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 = \\ x_4 = x_1 + x_2 + x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Show L.I.

$x_1 = c_1$

$x_2 = c_2$

$x_3 = c_3$

$x_4 = c_1 + c_2 + c_3$